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EXISTENCE OF A MILD SOLUTION TO A SECOND-ORDER IMPULSIVE FUNCTIONAL-DIFFERENTIAL EQUATION WITH A NONLOCAL CONDITION

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An abstract second-order semilinear functional-differential equation such that the linear part of its right-hand side is given by the infinitesimal generator of a strongly continuous cosine family of bounded linear operators, and provided with impulse and nonlocal conditions is studied. Under not too restrictive conditions the existence of a mild solution is proved using Schauder's fixed point theorem.

1. Introduction

Many evolutionary processes in nature are characterized by the fact that at certain instants of time they experience a rapid change of their states. The theory of the impulsive differential equations is one of the attractive branches of differential equations which has extensive realistic mathematical modelling applications in physics, chemistry, engineering, and biological and medical sciences. The nonlocal condition generalizes the classical initial condition. In our previous papers [1, 2] we found sufficient conditions for the existence, uniqueness and continuous dependence of a mild solution of a first-order impulsive functional-differential evolution nonlocal Cauchy problem such that the linear part of the right-hand side of the differential equation is given by the infinitesimal generator of a strongly continuous semigroup of bounded linear operators. In our recent paper [3] we proved theorems for existence and uniqueness of a mild and classical solution

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of an abstract second-order semilinear functional-differential equation such th
the linear part of its right-hand side is given by the infinitesimal generator o
strongly continuous cosine family of bounded linear operators, and provided wi
impulse and nonlocal conditions.

In the present paper we consider the abstract second-order nonlinear imp
sive functional-differential equation with nonlocal condition

$$
\begin{aligned}
(1)\qquad & x''(t) = Ax(t) \\
& +f(t, x(t), x(b_1(t)), \ldots, x(b_m(t)), x'(t), x'(b_1(t)), \ldots, x'(b_m(t))), \\
& t \in (0,T] \setminus \{\tau_1, \tau_2, \ldots, \tau_\kappa\}, \\
& \Delta x(\tau_k) = I_k(x(\tau_k)), \\
& \Delta x'(\tau_k) = \overline{I}_k(x(\tau_k), x'(\tau_k)), \quad k = \overline{1,\kappa}, \\
& x(0) = x_0, \\
& x'(0) = x_1 - g(x),
\end{aligned}
$$

where A is a linear operator from a real Banach space X with norm $\|\cdot\|$ into its
$x:\ [0,T] \to X$, $\Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0) \equiv x(\tau_k + 0) - x(\tau_k)$, $\Delta x'(\tau_k)$
$x'(\tau_k + 0) - x'(\tau_k - 0) \equiv x'(\tau_k + 0) - x'(\tau_k)$, $0 < \tau_1 < \tau_2 < \cdots < \tau_\kappa < T$ are t
instants of impulse effect, $f:\ [0,T] \times X^{2m+2} \to X$, $b_i:\ [0,T] \to [0,T]$ $(i = \overline{1,}$
$I_k:\ X \to X$, $\overline{I}_k:\ X^2 \to X$, $x_0, x_1 \in X$, and $g(x)$ is a function with values in
to be specified later.

Next we shall need the following definitions [8, 6, 7, 9].

Definition 1. *A one-parameter family* $\{C(t)\ :\ t \in \mathbb{R}\}$ *of bounded lin operators mapping the Banach space* X *into itself is called a* strongly continu cosine family *if and only if*

1) $C(s+t) + C(s-t) = 2C(s)C(t)$ *for all* $s, t \in \mathbb{R}$.

2) $C(0) = I$ *(the identity operator).*

3) $C(t)x$ *is continuous in* t *on* $\mathbb{R}$ *for each fixed* $x \in X$.

Definition 2. *The infinitesimal generator of a strongly continuous cos family* $\{C(t)\}$ *is the operator* $A:\ X \supset \mathcal{D}(A) \to X$ *defined by*

$$Ax := \frac{d^2}{dt^2}C(t)x\Big|_{t=0}, \quad x \in \mathcal{D}(A),$$

where

$$\mathcal{D}(A) := \left\{x \in X\ :\ C(t)x \ \text{ is of class } \ C^2 \ \text{ with respect to } \ t\right\}.$$

Example 1. If A is a positive bounded operator in a Hilbert space X, then $\{\cosh(\sqrt{A}t)\}$ is a strongly continuous cosine family of operators with infinitesimal generator A. If A is a negative bounded operator in a Hilbert space X, then $\{\cos(\sqrt{-A}t)\}$ is a strongly continuous cosine family of operators with infinitesimal generator A.

We introduce the assumptions:

A1 The operator A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators from X to itself.

A2 The adjoint operator A^* is densely defined in X^*, *i.e.*, $\overline{\mathcal{D}(A^*)} = X^*$.

Let us denote

$$E := \{x \in X : \ C(t)x \ \text{ is of class } \ C^1 \ \text{ with respect to } \ t\}.$$

The associated sine family $\{S(t) : t \in \mathbb{R}\}$ is defined by

$$S(t)x := \int_0^t C(s)x\,ds, \qquad x \in X, \quad t \in \mathbb{R}.$$

Further on we denote by $\|C(t)\|$, $\|S(t)\|$ and $\|A\|$ the operator norms of $C(t)$, $S(t)$ and A in the Banach space X, respectively. From Assumption **A1** it follows that there is a constant $M \geq 1$ such that

$$\|C(t)\| \leq M \quad \text{and} \quad \|S(t)\| \leq M \quad \text{for} \quad t \in [0,T]. \tag{2}$$

Following [5], we present a result obtained by J. Bochenek in [4].

Let us consider the Cauchy problem

$$\begin{aligned} x''(t) &= Ax(t) + h(t), \qquad t \in (0,T], \\ & x(0) = x_0, \quad x'(0) = x_1. \end{aligned} \tag{3}$$

Definition 3. *A function $x : [0,T] \to X$ is said to be a classical solution of problem* (3) *if*

$$x \in C^1([0,T],X) \cap C^2((0,T],X),$$
$$x(0) = x_0 \quad \textit{and} \quad x'(0) = x_1,$$
$$x''(t) = Ax(t) + h(t) \quad \textit{for} \quad t \in (0,T].$$

Theorem 1. *Suppose that*
1) *Assumptions* **A1** *and* **A2** *are satisfied;*
2) $h: [0,T] \to X$ *is Lipschitz continuous;*
3) $x_0 \in \mathcal{D}(A)$ *and* $x_1 \in E$.
Then problem (3) *has a unique classical solution given by the formula*

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s)\,ds, \quad t \in [0,T].$$

It is easy to see that this result can be generalized for the impulsive system

$$\begin{aligned} x''(t) &= Ax(t) + h(t), \quad t \in (0,T] \setminus \{\tau_1, \tau_2, \dots, \tau_\kappa\}, \qquad (4)\\ \Delta x(\tau_k) &= I_k, \\ \Delta x'(\tau_k) &= \bar{I}_k, \quad k = \overline{1,\kappa}, \qquad (5)\\ x(0) &= x_0, \quad x'(0) = x_1. \qquad (6) \end{aligned}$$

For convenience we denote $J = [0,T]$, $J_0 = [0,\tau_1]$, $J_k = (\tau_k, \tau_{k+1}]$, $k = \overline{1, \kappa-1}$, $J_\kappa = (\tau_\kappa, T]$, $J' = J \setminus \{0, \tau_1, \tau_2, \dots, \tau_\kappa\}$. For a function $x: J \to X$ we denote by x_k the restriction of x to J_k, $k = \overline{0,\kappa}$, with $\|x_k\|_{J_k} = \sup_{s \in J_k} \|x_k(s)\|$. If the function x_k happens to be differentiable, we denote its derivative by x_k'. Further we define the following classes of functions:

$$\begin{aligned} PC(J,X) &= \{x: J \to X \mid x_k \in C(J_k, X),\ k = \overline{0,\kappa}, \\ &\qquad \text{and there exist } x(\tau_k + 0),\ k = \overline{1,\kappa}\}, \\ PC^1(J,X) &= \{x \in PC(J,X) \mid x_k' \in C(J_k, X),\ k = \overline{0,\kappa}, \\ &\qquad \text{and there exist } x'(\tau_k + 0),\ k = \overline{1,\kappa}\}. \end{aligned}$$

$PC(J,X)$ is a Banach space with norm $\|x\|_{PC} = \max\{\|x_k\|_{J_k},\ k = \overline{0,\kappa}\}$, and $PC^1(J,X)$ is a Banach space with norm $\|x\|_{PC^1} = \|x\|_{PC} + \|x'\|_{PC}$.

Definition 4. *A function* $x \in PC^1(J,X) \cap C^2(J',X)$ *is called a classical solution of problem* (4)–(6) *if it satisfies the differential equation* (4) *on* J' *together with the impulse conditions* (5) *and the initial conditions* (6).

Theorem 2. [3] *Suppose that*
1) *Assumptions* **A1** *and* **A2** *are satisfied;*
2) $h \in PC(J,X)$ *is such that its restrictions to* J_k *are Lipschitz continuous,* $k = \overline{0,\kappa}$;

3) $x_0 \in \mathcal{D}(A)$ *and* $x_1 \in E$.
4) $I_k \in \mathcal{D}(A)$ *and* $\overline{I}_k \in E$ *for* $k = \overline{1, \kappa}$.
Then problem (4)–(6) *has a unique classical solution given by the formula*

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s)\,ds$$
$$+ \sum_{0<\tau_k<t} C(t-\tau_k)I_k + \sum_{0<\tau_k<t} S(t-\tau_k)\overline{I}_k, \quad t \in J.$$

Theorem 2 can be proved by applying Theorem 1 on each interval of continuity J_k, $k = \overline{0, \kappa}$.

This theorem suggests the following definition.

Definition 5. *A function* $x \in PC^1(J, X)$ *satisfying the integro-summary equation*

$$(7)\quad x(t) = C(t)x_0 + S(t)(x_1 - g(x))$$
$$+ \int_0^t S(t-s)f(s, x(s), x(b_1(s)), \dots, x(b_m(s)), x'(s), x'(b_1(s)), \dots, x'(b_m(s)))\,ds$$
$$+ \sum_{0<\tau_k<t} C(t-\tau_k)I_k(x(\tau_k)) + \sum_{0<\tau_k<t} S(t-\tau_k)\overline{I}_k(x(\tau_k), x'(\tau_k)), \quad t \in J.$$

is said to be a mild solution of the nonlocal problem (1).

In the next section we present not too restrictive sufficient conditions for the nonlocal problem (1) to have a mild solution.

2. Main result

For $r > 0$ denote

$$X_r = \left\{u \in PC^1(J, X) : \|u\|_{PC^1} \le r\right\}.$$

Next we introduce the assumption:

A3 $x_0 \in E$, $x_1 \in X$, $f : J \times X^{2m+2} \to X$ is such that the function $t \mapsto f(t, u_0, u_1, \dots, u_m, v_0, v_1, \dots, v_m)$ belongs to $PC(J, X)$ for each $(u_0, u_1, \dots, u_m, v_0, v_1, \dots, v_m) \in X^{2m+2}$ and the function $(u_0, u_1, \dots, u_m, v_0, v_1, \dots, v_m) \mapsto f(t, u_0, u_1, \dots, u_m, v_0, v_1, \dots, v_m)$ belongs to $C(X^{2m+2}, X)$ for each $t \in J$, $b_i \in C(J, J)$ $(i = \overline{1, m})$, $I_k \in C(X, E)$,

$\overline{I}_k \in C(X^2, X)$, $g \in C(PC^1(J,X),X)$ and there exist positive constants C_i $(i = \overline{1,4})$ and r such that

$$\begin{cases} \|f(t,u_0,u_1,\dots,u_m,v_0,v_1,\dots,v_m)\| \le C_1 \\ \qquad\qquad\qquad\qquad \text{for } t \in J,\ \|u_i\| + \|v_i\| \le r\ (i = \overline{0,m}), \\ \|g(u)\| \le C_2 \text{ for } u \in X_r, \\ \|I_k(u)\| \le C_3 \text{ for } \|u\| \le r,\ k = \overline{1,\kappa}, \\ \|\overline{I}_k(u,v)\| \le C_4 \text{ for } \|u\| + \|v\| \le r,\ k = \overline{1,\kappa} \end{cases}$$

and

$$r \ge M\left\{(1+\|A\|)\|x_0\| + 2\|x_1\| + 2TC_1 + 2C_2 + \kappa[(1+\|A\|)C_3 + 2C_4]\right\},$$

where M is the constant from inequalities (2).

Example 2. Consider the scalar problem

$$\begin{aligned} (8)\quad x''(t) &= x(t) + C_1\left[(x(t)+x'(t))^2/2 + \sum_{i=1}^{m} 2^{-i-1}(x(b_i(t)) + x'(b_i(t)))^2\right], \\ &\qquad t \in (0,2] \setminus \{\tau_1, \tau_2, \dots, \tau_\kappa\}, \\ \Delta x(\tau_k) &= c_k x(\tau_k), \quad k = \overline{1,\kappa}, \\ \Delta x'(\tau_k) &= c'_k\left(x(\tau_k) + x'(\tau_k)\right), \quad k = \overline{1,\kappa}, \\ x(0) &= x_0, \\ x'(0) &= x_1 - C_2 \sum_{j=1}^{p} 2^{-j}\left(x(t_j) + x'(t_j)\right), \end{aligned}$$

where $0 < \tau_1 < \dots < \tau_\kappa < 2$, $0 < t_1 < \dots < t_p \le 2$ and

$$|c_k| \le C_3, \quad |c'_k| \le C_4 \quad \text{for} \quad k = \overline{1,\kappa}.$$

Here $X = \mathbb{R}$, $A = 1$, $C(t) = \cosh t$, $S(t) = \sinh t$, $\max\limits_{0\le t\le 2} \cosh t = \cosh 2 < 4$ since $\operatorname{arcosh} 4 = 2.063437$, $\max\limits_{0\le t\le 2} \sinh t = \sinh 2 < \cosh 2 < 4$, thus we can choose $M = 4$. If we take $r = 1$, it is easy to see that the constants C_i, $i = \overline{1,4}$ in problem (8) can be chosen as the constants in condition **A3**. Then condition **A3** is satisfied if

$$\|x_0\| + \|x_1\| + 2C_1 + C_2 + \kappa(C_3 + C_4) \le \frac{1}{8}.$$

Now we formulate our main result.

Theorem 3. *Suppose that assumptions* **A1** *and* **A3** *are satisfied. Then problem* (1) *has a mild solution.*

Proof. We can write equation (7) in an operator form

$$x = \mathcal{F}x,$$

where the operator $\mathcal{F}$ is defined on the Banach space $PC^1(J,X)$ by the formula

$$\begin{aligned}(9)\quad (\mathcal{F}u)(t) &= C(t)x_0 + S(t)(x_1 - g(u))\\ &+ \int_0^t S(t-s)f(s,u(s),u(b_1(s)),\dots,u(b_m(s)),u'(s),u'(b_1(s)),\dots,u'(b_m(s)))\,ds\\ &+ \sum_{0<\tau_k<t} C(t-\tau_k)I_k(u(\tau_k)) + \sum_{0<\tau_k<t} S(t-\tau_k)\overline{I}_k(u(\tau_k),u'(\tau_k)), \quad t\in J.\end{aligned}$$

It is easy to see that $\mathcal{F}u \in PC(J,X)$ for $u\in PC^1(J,X)$. Moreover, since

$$\begin{aligned}(10)\qquad \frac{d}{dt}(\mathcal{F}u)(t) &= AS(t)x_0 + C(t)(x_1 - g(u))\\ &+ \int_0^t C(t-s)f(s,u(s),u(b_1(s)),\dots,u(b_m(s)),u'(s),u'(b_1(s)),\dots,u'(b_m(s)))\,ds\\ &+ \sum_{0<\tau_k<t} AS(t-\tau_k)I_k(u(\tau_k)) + \sum_{0<\tau_k<t} C(t-\tau_k)\overline{I}_k(u(\tau_k),u'(\tau_k)), \quad t\in J,\end{aligned}$$

we have $\frac{d}{dt}\mathcal{F}u \in PC(J,X)$ and, consequently, $\mathcal{F}u\in PC^1(J,X)$. Thus the operator $\mathcal{F}$ maps the Banach space $PC^1(J,X)$ into itself.

In order to find a mild solution x of problem (1) we are looking for a fixed point of the operator $\mathcal{F}$ in a suitably chosen subset of the Banach space $PC^1(J,X)$. Obviously, X_r is a convex subset of $PC^1(J,X)$. To apply Schauder's fixed point theorem, we shall show that $\mathcal{F}(X_r)$ is a precompact subset of X_r.

Let $u\in X_r$. Then from (9) by virtue of assumption **A3** we obtain

$$\begin{aligned}\|\mathcal{F}u\|_{PC} &\le \max_{t\in J}\|C(t)\|\cdot\|x_0\| + \max_{t\in J}\|S(t)\|(\|x_1\| + \|g(u)\|)\\ &+ \max_{t\in J}\int_0^t \|S(t-s)\|\cdot\|f(s,u(s),u(b_1(s)),\dots,u(b_m(s)),u'(s),\\ &\qquad\qquad u'(b_1(s)),\dots,u'(b_m(s)))\|\,ds\\ &+ \max_{t\in J}\sum_{0<\tau_k<t}\|C(t-\tau_k)\|\cdot\|I_k(u(\tau_k))\|\end{aligned}$$

$$
\begin{aligned}
&+\max_{t\in J}\sum_{0<\tau_k<t}\|S(t-\tau_k)\|\cdot\|\overline{I}_k(u(\tau_k),u'(\tau_k))\| \\
&\le M\left[\|x_0\|+\|x_1\|+C_2+TC_1+\kappa(C_3+C_4)\right].
\end{aligned}
$$

Similarly, from (10) by virtue of assumption **A3** we obtain

$$
\left\|\frac{d}{dt}\mathcal{F}u\right\|_{PC} \le M\left[\|A\|\cdot\|x_0\|+\|x_1\|+C_2+TC_1+\kappa(\|A\|C_3+C_4)\right],
$$

and thus

$$
\begin{aligned}
\|\mathcal{F}u\|_{PC^1} &= \|\mathcal{F}u\|_{PC}+\left\|\frac{d}{dt}\mathcal{F}u\right\|_{PC} \\
&\le M\left\{(1+\|A\|)\|x_0\|+2\|x_1\|+2C_2+2TC_1+\kappa[(1+\|A\|)C_3+2C_4]\right\}\le r.
\end{aligned}
$$

The last inequality shows that $\mathcal{F}u\in X_r$ for $u\in X_r$, that is, $\mathcal{F}(X_r)\subset X_r$.

It also implies that the family of functions $\{\mathcal{F}u:\ u\in X_r\}$ is equibounded. Finally, we shall show that it is also equicontinuous on each interval of continuity J_k, $k=\overline{0,\kappa}$.

Let $u\in X_r$ and $0\le t_1<t_2\le T$. Then we have

$$
\begin{aligned}
&(\mathcal{F}u)(t_2)-(\mathcal{F}u)(t_1) = (C(t_2)-C(t_1))\,x_0+(S(t_2)-S(t_1))\,(x_1-g(u)) \\
&+\int_0^{t_1}(S(t_2-s)-S(t_1-s))\,f(s,u(s),u(b_1(s)),\dots,u(b_m(s)), \\
&\qquad\qquad u'(s),u'(b_1(s)),\dots,u'(b_m(s)))\,ds \\
&+\int_{t_1}^{t_2}S(t_2-s)f(s,u(s),u(b_1(s)),\dots,u(b_m(s)),u'(s),u'(b_1(s)),\dots,u'(b_m(s)))\,ds \\
&+\sum_{0<\tau_k<t_1}(C(t_2-\tau_k)-C(t_1-\tau_k))\,I_k(u(\tau_k))+\sum_{t_1\le\tau_k<t_2}C(t_2-\tau_k)I_k(u(\tau_k)) \\
&+\sum_{0<\tau_k<t_1}(S(t_2-\tau_k)-S(t_1-\tau_k))\,\overline{I}_k(u(\tau_k),u'(\tau_k)) \\
&+\sum_{t_1\le\tau_k<t_2}S(t_2-\tau_k)\overline{I}_k(u(\tau_k),u'(\tau_k)).
\end{aligned}
$$

From here by virtue of assumption **A3** we derive the inequality

$$
\begin{aligned}
&\|(\mathcal{F}u)(t_2)-(\mathcal{F}u)(t_1)\|\le\|C(t_2)-C(t_1)\|\cdot\|x_0\|+\|S(t_2)-S(t_1)\|(\|x_1\|+C \\
&+\int_0^{t_1}\|S(t_2-s)-S(t_1-s)\|\,ds\cdot C_1+(t_2-t_1)\,MC_1
\end{aligned}
$$

$$+ \sum_{0<\tau_k<t_1} \|C(t_2-\tau_k) - C(t_1-\tau_k)\|C_3 + i(t_1,t_2)MC_3$$
$$+ \sum_{0<\tau_k<t_1} \|S(t_2-\tau_k) - S(t_1-\tau_k)\|C_4 + i(t_1,t_2)MC_4,$$

where $i(t_1,t_2)$ is the number of instants of impulse effect in the interval $[t_1,t_2)$.

We first notice that the right-hand side of this inequality is independent of the choice of $u \in X_r$. Further on, due to the uniform continuity of the functions $C(t)$ and $S(t)$ on J in the operator norm, all norms in the right-hand side of this inequality can be made uniformly small for sufficiently small $t_2 - t_1$. Finally, $i(t_1,t_2)$ is zero for t_1, t_2 both in one of the intervals of continuity J_k, $k = \overline{0,\kappa}$, and a positive integer otherwise. This shows that the family of functions $\{\mathcal{F}u : u \in X_r\}$ is equicontinuous on each interval of continuity. Since all assumptions of Arzela-Ascoli's theorem are satisfied on each interval of continuity, the set $\mathcal{F}(X_r)$ is a precompact subset of X_r. According to Schauder's fixed point theorem, the operator $\mathcal{F}$ has a fixed point x in X_r, which is a mild solution of problem (1). □

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